Analysis Qualifying Exam Fall 2007

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. (a) Prove that an uncountable subset of $\mathbb{R}$ must have an uncountable bounded subset.
(b) Use part (a) to deduce that every uncountable subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.

2. Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0), s_2 = f(s_1)$, etc. Prove that $\{s_n\}$ is a convergent sequence.

3. (a) State a definition of *Riemann integrable* that makes use of partitions.
(b) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. Suppose f is Riemann integrable on $[a, c]$ and on $[c, b]$. Using your definition from part (a), show f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

4. Suppose a_n and b_n are nonnegative for all $n \in \mathbb{N}$.
(a) Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ also converges.
(b) Improve the result of part (a) by replacing the requirement of convergence of $\sum_{n=1}^{\infty} b_n$ with something weaker.

5.(a) Give an explicit description of $D = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{ne^{nx}}{5^n} \text{ converges}\}$.
(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{ne^{nx}}{5^n}$ for $x \in D$. For which $x \in D$ is f continuous?