

Analysis Qualifying Exam

Fall 2006

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Let $A \subseteq \mathbb{R}$. Prove that the set of limit points of A is closed.
2. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers. Suppose f is a differentiable function on $[0, \infty)$ with $f(0) = 0$ and f' bounded. Prove that

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ converges, then } \sum_{n=1}^{\infty} f(a_n) \text{ converges absolutely.}$$

3. For each natural number n , let $f_n(x) = nx^n(1-x)$.

(a) Find the pointwise limit of $\{f_n\}$ on $[0, 1]$.

(b) Does

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx?$$

(c) Show that $\{f_n\}$ does not converge uniformly on $[0, 1]$.

$$\text{(Note : } \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{n+1} = e^{-1} \text{.)}$$

4. (a) State a definition of the Riemann integral.

(b) Assume g and f are nonnegative functions defined on $[a, b]$. Suppose f is bounded and Riemann integrable on $[a, b]$, and g is continuous on $[a, b]$. Prove directly from the definition of the Riemann integral that fg is Riemann integrable on $[a, b]$.

5. (a) Define uniform continuity of a function f on a set A .

(b) Prove that $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on \mathbb{R} .