Analysis Qualifying Exam Fall 2005

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{R}$. Assume that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ for each natural number n. Show that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

2. Let f be a function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Show that if $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0,1)$, then $f(x) = 0$ for all $x \in [0,1]$.

3. Let f, g, and h be bounded real-valued functions on the closed interval $[0,1]$. Suppose that f and h are Riemann integrable on $[0,1]$, $\int_0^1 f(x)\,dx = \int_0^1 h(x)\,dx$, and $f(x) \leq g(x) \leq h(x)$ for all $x \in [0,1]$. Show that g is Riemann integrable on $[0,1]$.

4. Let $f : [a,b] \to \mathbb{R}$ be monotonically decreasing. Show that

$$\lim_{x \to c^+} f(x)$$

exists for each $c \in (a,b)$.

5. Show that

$$\int_0^{\pi} \left[\sum_{n=1}^{\infty} \frac{n \cos(\frac{nx}{2})}{\pi^{2n}} \right] dx = \frac{2\pi^2}{\pi^4 + 1} \, .$$

Justify each step in your solution.