# Analysis Qualifying Exam — Fall 2004

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. (a) Given a set $S \subseteq \mathbb{R}$, define what it means for a point $p$ to be an interior point of $S$.

   (b) For $S \subseteq \mathbb{R}$, let $S^o = \{p \in \mathbb{R} : p \text{ is an interior point of } S\}$. Show that $S^o$ is an open subset of $S$.

2. Let $\{a_n\}$ be a bounded sequence and set $B = \{p \in \mathbb{R} : p \leq a_n \text{ for infinitely many } n\}$. Show that $B$ is nonempty, bounded above and that there exists a subsequence $\{a_{n_k}\}$ that converges to the least upper bound of $B$.

3. Let $f : [a, b] \to \mathbb{R}$ be bounded and let $\mathcal{P} = \{a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b\}$ be a partition of $[a, b]$.

   (a) Define the upper and lower Riemann sums for $f$ with this partition.

   (b) Given $f : [a, b] \to \mathbb{R}$, characterize Riemann integrability of $f$ in terms of upper and lower Riemann sums.

   (c) Show that if $f$ is Riemann integrable on $[a, b]$, then $|f|$ is Riemann integrable on $[a, b]$ as well.

4. Suppose $f$ is continuous $[a, b]$. Further suppose that $f$ is differentiable at all points of $(a, b)$ except possibly at a single point $x_0 \in (a, b)$. If $\lim_{x \to x_0} f'(x)$ exists, show that $f'(x_0)$ exists and $f'(x_0) = \lim_{x \to x_0} f'(x)$.

5. Let $f_j : (0, \infty) \to \mathbb{R}$ be defined for each $j = 1, 2, 3, \ldots$ by

$$f_j(x) = \frac{\sin jx}{j^3 \sqrt{x}}$$

   (a) Find the pointwise limit of $\{f_j\}$ and determine whether or not the convergence is uniform.

   (b) Show that the series $\sum_{j=1}^{\infty} f_j(x)$ converges pointwise to a function $f$ on an interval containing $\pi$.

   (c) Prove that $f$ is differentiable at $\pi$ and compute $f'(\pi)$. (You may leave your answer in the form of a sum.)