Analysis Qualifying Exam Spring 2008

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Suppose K is a compact subset of $\mathbf{R}$ and $f : K \to (0, \infty)$ is a continuous function. Show there exists $\delta > 0$ such that for all $x \in K$, $f(x) \geq \delta$.

2. Let f be a twice differentiable real-valued function defined on the interval (a, b). Suppose that $a < x_1 < x_2 < x_3 < b$, $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove there exists $c \in (a, b)$ such that $f''(c) > 0$.

3. (a) State a definition of what it means for a a function f to be Riemann integrable on a finite interval $[a, b]$.
(b) Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $f(x) = 0$ for irrational numbers x and

$$f\left(\frac{m}{n}\right) = \begin{cases} 7 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd} \end{cases}$$

for relatively prime integers m and n. Use the definition of Riemann integrable to prove that f is not Riemann integrable on $[0, 1]$.

4. (a) Suppose a_n is nonnegative for all $n \in \mathbf{N}$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.
(b) Prove or give a counter-example to the following statement.

> If a_n is a real number for all $n \in \mathbf{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions defined on an interval $[a, b]$ and f be a real-valued function defined on $[a, b]$.

(a) Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

(b) Does part (a) remain true if we replace "converges uniformly" with "converges pointwise"? Prove or give a counterexample.