Analysis Qualifying Exam Spring 2005

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and $c \in (a, b)$. If $f'(c) \neq 0$, show that there exists $\delta > 0$ such that $f(x) \neq f(c)$ for all $x \in (c - \delta, c + \delta)$.

2. (a) Let $S \subseteq \mathbb{R}$ be an arbitrary set and let $\varepsilon > 0$. Show that the set
$$T = \{t \in \mathbb{R} : |t - s| < \varepsilon \text{ for some } s \in S\}$$
is open.

   (b) Use a definition of connectedness to show that no subset of $\mathbb{Q}$ is connected in the real line.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and nonnegative. Show that
$$\lim_{n\to\infty} \left[\int_a^b [f(x)]^n \, dx\right]^{\frac{1}{n}} = \max\{f(x) : x \in [a, b]\}$$

4. Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable.

   (a) Show that $\int_a^b [f(x)]^2 \, dx$ exists.

   (b) Use the **result** of part (a) to show that if f and g are both Riemann integrable on $[a, b]$, then so is fg.

5. (a) Give an example of a sequence of everywhere-differentiable functions $\{f_n\}$ that converges pointwise to a discontinuous limit function f on some closed interval.

(b) Give an example of a sequence of everywhere-differentiable functions $\{f_n\}$ such that $\sum f_n$ converges pointwise but not uniformly on some bounded, open interval (a, b).

(c) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined for each $n = 1, 2, 3, \ldots$ by

$$f_n(x) = \begin{cases} -x & \text{if } x < -\frac{1}{n} \\ \frac{1}{2n}(n^2x^2 + 1) & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ x & \text{if } x > \frac{1}{n} \end{cases}$$

(i) Find the pointwise limit of $\{f_n\}$.
(ii) Prove that $\{f_n\}$ converges uniformly to its pointwise limit.
(iii) Does $\{f_n'\}$ converge uniformly to its pointwise limit? Justify your answer.