Analysis Qualifying Exam Summer 2006

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Suppose there exists $\delta > 0$ such that for all $x, y \in A$ with $x \neq y$ we have $|x - y| \geq \delta$. Prove that A contains its supremum.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that for all $n \in \mathbb{N}$,

$$y_n \neq 0 \text{ and } \frac{x_n}{y_n} \to 1.$$

Show that if either sequence is bounded, then $(x_n - y_n) \to 0$.

3. Let $\{f_n\}$ be a decreasing sequence of nonnegative, continuous functions on a closed, bounded set S. Show that if $f_n \to 0$ pointwise on S, then $f_n \to 0$ uniformly on S. You may not use Dini's Theorem.

4. Suppose f is bounded and Riemann integrable on $[a, b]$. Without appealing to measure theory, prove that $|f|$ is Riemann integrable on $[a, b]$.

5. Find the smallest natural number R that makes the following assertion true:

"For every infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ having R distinct real roots, the fifth derivative, $f^{(5)}$, has at least one real root."

Provide a proof, including an example to show your choice for R is the smallest.