Analysis Qualifying Exam Winter 2008

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Let $E \subset \mathbf{R}$ be a nonempty bounded set.
(a) Define $\sup(E)$.
(b) Assume $\sup(E) \notin E$. Prove that there exists a strictly increasing sequence (x_n) which converges to $\sup(E)$ and $x_n \in E$ for each $n \in \mathbf{N}$.

2. Let $f : [0, \infty) \to \mathbf{R}$ be a function which is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Suppose that $f(0) = 0$ and f' is increasing. Let $g(x) = f(x)/x$ for $x > 0$. Prove that g is increasing. Do not assume that f is twice differentiable.

3. Classify all functions $f : [0, \infty) \to [0, \infty)$ which satisfy the following three conditions: (1) f is Riemann integrable on any closed and bounded subinterval of $[0, \infty)$, (2) $f(x) > 0$ for all $x > 0$ and (3) $(f(x))^2 = \int_0^x f(t)\, dt$ for all $x > 0$.
Hint: First prove that any such f must be differentiable.

4. Let (x_n) be a sequence of positive numbers. Prove that $\sum_{n=1}^{\infty} \frac{x_n}{1 + x_n}$ is convergent if and only if $\sum_{n=1}^{\infty} x_n$ is convergent. (You may use the well-known tests for convergence.)

5. Let $f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^3}$.

(a) Prove that f is Riemann integrable on $[0, \frac{\pi}{2}]$ and $\int_0^{\frac{\pi}{2}} f(x)\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4}$.

(b) Prove that f is differentiable.