Analysis Qualifying Exam Winter 2006

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. For every $n \in \mathbb{N}$, let $A_n \subseteq \mathbb{R}$. Is the closure of the intersection of these sets equal to the intersection of their closures?

$$\text{i.e. Is } \overline{\left[\bigcap_{n=1}^{\infty} A_n\right]} = \bigcap_{n=1}^{\infty} \overline{A_n} \quad ?$$

If not, state the exact containment relationship between the two sets, and provide a counterexample to show equality need not hold.

2. Consider the function g defined by

$$g(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} .$$

(a) Show that g is differentiable at $x = 0$.
(b) Find $g'(0)$.
(c) Prove or disprove: If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(c) \neq 0$, then f is monotone on an interval containing c.

3. Let $\{a_j\} \subseteq \mathbb{R}$ such that $\sum_{j=1}^{\infty} a_j = \frac{3\pi}{4}$. For every $n \in \mathbb{N}$, define

$$T_n = \frac{1}{n} \sum_{j=1}^{n} S_j \text{ where for each } j \in \mathbb{N}, S_j = \sum_{k=1}^{j} a_k.$$

Does $\{T_n\}_{n=1}^{\infty}$ converge? If so, find its sum.

4. Suppose that f is bounded and Riemann integrable on an interval $[a, b]$ and that g is uniformly continuous on the range of f. Without appealing to measure theory, prove that the composition $g \circ f$ is Riemann integrable on $[a, b]$.

5. Let $f : [0, 1] \to \mathbb{R}$ be non-negative and continuous. Show that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0).$$