

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. (a) Let $S \subseteq \mathbb{R}$ be a bounded set. Give the precise definition of the infimum (g.l.b.) and supremum (l.u.b.) of the set S .
(b) For $S \subseteq \mathbb{R}$, let $|S| = \{|s| : s \in S\}$. If $\sup S = \alpha$ and $\inf S = \beta$, find $\sup |S|$ and $\inf |S|$ in terms of α and β . (Are there cases to consider here?)
2. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *Lipschitz of order* $\alpha > 0$ if
$$|f(x) - f(y)| \leq |x - y|^\alpha \text{ for all } x, y \in \mathbb{R}$$
 - (a) Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz of order 2, then f is constant.
 - (b) Classify the values of $\alpha > 0$ such that the implication
$$f \text{ Lipschitz of order } \alpha \Rightarrow f \text{ is constant}$$
remains valid (part (a) shows that 2 is one such value of α).
 - (c) Give a concrete example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a value of $\alpha > 0$ such that f is Lipschitz of order α yet not differentiable.
3. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded.
 - (a) Define what it means for f to be Riemann integrable in terms of upper and lower sums.
 - (b) Show that if $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$.
4. Let $\{a_n\}$ be a sequence of real numbers.
 - (a) Define what it means for $\{a_n\}$ to converge to a limit α .
 - (b) Suppose $a_n \rightarrow \alpha$, $b_n \rightarrow \beta$, $b_n \neq 0 \forall n$, and $\beta \neq 0$. Show that $a_n/b_n \rightarrow \alpha/\beta$.

5. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined for each $n = 1, 2, 3, \dots$ by

$$f_n(x) = \frac{nx^2 e^{-\frac{x^2}{n}}}{1 + nx^2}$$

- (a) Find the pointwise limit of the sequence $\{f_n\}$.
- (b) A quick sketch should convince you that $1 - e^{-t} \leq t$ for all $t \geq 0$. Use this inequality to help you show that the sequence $\{f_n\}$ converges uniformly on the interval $[1, 2]$.
- (c) Does $\{f_n\}$ converge uniformly on any open interval containing 0? Explain your reasoning.
- (d) Compute $f_n(\sqrt{n})$. Does this tell you anything about whether the sequence $\{f_n\}$ converges uniformly on the interval $[1, \infty)$? Explain.