Algebra Qualifying Exam
September 16, 2008

Do all five problems.

1. Let G be a group and suppose that G has exactly one subgroup H of order 50. Prove that H is a normal subgroup of G.

2. Let $\mathbb{R}_n[x]$ be the $\mathbb{R}$-vector space of polynomials of degree less than or equal to n with coefficients in $\mathbb{R}$ and let $D : \mathbb{R}_n[x] \longrightarrow \mathbb{R}_n[x]$ be the derivative operator.

   (a) Find the matrix of D with respect to the standard ordered basis $\{1, x, x^2, \ldots, x^n\}$.

   (b) Prove that D is nilpotent (i. e., there exists a positive integer m such that $D^m(p(x)) = 0$ for all $p(x) \in \mathbb{R}_n[x]$).

3. Prove that a finite integral domain is a field.

4. Let $P(\mathbb{R})$ be the vector space of all polynomials with real coefficients and let W be the subspace of $P(\mathbb{R})$ defined by

$$W = \{p(x) \in P(\mathbb{R}) \,|\, \deg(p(x)) \leq 1\}.$$

   Find the orthogonal projection of the polynomial $h(x) = 4 + 3x - 2x^2$ onto W with respect to the inner product on $P(\mathbb{R})$ given by

$$\langle p(x),\, q(x) \rangle = \int_0^1 f(t)g(t)\, dt.$$

5. Let G be a group and let H, K be normal subgroups of G with $H \leq K$.

   (a) Prove that $K/H \trianglelefteq G/H$.

   (b) Prove that $(G/H)/(K/H) \cong G/K$.