Algebra Qualifying Exam

September 11, 2007

Do all five problems.

1. Let K and N be normal subgroups of a group G. Prove that if $K \cap N = \{e\}$, then $kn = nk$ for all $k \in K$ and $n \in N$.

2. Let $\mathcal{P}_2$ be the vector space of all polynomials in $\mathbb{R}[\,x\,]$ of degree less than or equal to two. Define a linear transformation $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ by $T(f) = f' + f''$ for all $f \in \mathcal{P}_2$.

   (a) Find the matrix of T with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$.

   (b) Determine whether T is diagonalizable or not.

3. Let S be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that the set $A = \{f \in S \,|\, f(0) = 0\}$ is a maximal ideal of S.

4. Let H be a subgroup of a group G and let

$$C(H) = \{g \in G \,|\, gh = hg,\ \forall h \in H\}$$

   and

$$N(H) = \{x \in G \,|\, xHx^{-1} = H\}$$

   denote the **centralizer** and **normalizer** of H, respectively.

   (a) Prove that $C(H)$ is a subgroup of G.

   (b) Prove that $N(H)$ is a subgroup of G.

   (c) Prove that the quotient $N(H)/C(H)$ is isomorphic to a subgroup of $Aut(H)$. *Hint*: Show that, for each $g \in N(H)$, the function ϕ_g that conjugates every element of H by g is an element of $Aut(H)$.

5. Let A be an $n \times n$ matrix with entries in $\mathbb{R}$. If the characteristic polynomial of A factors as $(\lambda - 4)^3(\lambda + 3)^2$, list the possible Jordan canonical forms for A.