ALGEBRA QUALIFYING EXAM
September 2006

Do all five problems.

1. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the set

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} \right\}$$

   (with respect to the standard inner product on $\mathbb{R}^4$).

2. Suppose that λ_1 and λ_2 are distinct eigenvalues for the matrix A. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to λ_1 and λ_2 respectively, prove that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

3. Let G be an Abelian group and let $T(G) = \{g \in G \mid \text{order of } g < \infty\}$.
   (a) Prove that $T(G)$ is a subgroup of G.
   (b) Prove that every nontrivial element of $G/T(G)$ has infinite order.

4. Prove that a group G is Abelian if and only if the function $f : G \to G$ given by the rule $f(a) = a^{-1}$ is a homomorphism.

5. Let $I = \{a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Q}[x] \mid n \geq 0,\ a_0 + a_1 + \cdots + a_n = 0\}$.
   (a) Prove that I is an ideal of $\mathbb{Q}[x]$. (Hint: the map $\phi : \mathbb{Q}[x] \to \mathbb{Q}$ given by $\phi(f(x)) = f(1)$ is a ring homomorphism. You may use this fact without proof).
   (b) Prove that $\mathbb{Q}[x]/I$ is a field.