

ALGEBRA QUALIFYING EXAM
September 12, 2005

Do all five problems.

1. Let A be an $n \times n$ diagonalizable matrix with entries in a field F . Prove that if A has only one eigenvalue λ , then $A = \lambda I$.

2. Let V and W be finite dimensional vector spaces and let $T : V \rightarrow W$ be a linear transformation. Prove that

$$\dim(V) = \text{Rank}(T) + \text{Nullity}(T).$$

3. Let R be a commutative ring with identity. Prove that R is an integral domain if and only if, for all $x, y, z \in R$ with $x \neq 0$, $xy = xz$ implies $y = z$.

4. Given a group G , let G' be the subgroup of G generated by the set $\{x^{-1}y^{-1}xy \mid x, y \in G\}$.

- (a) Prove that G' is a normal subgroup of G .
- (b) Prove that G/G' is Abelian.

5. List all Abelian groups of order 144 (up to isomorphism).