ALGEBRA QUALIFYING EXAM
September 2004

Do all five problems.

(1) Let V be the inner product space of polynomials over $\mathbb{R}$, with $\langle f, g\rangle = \int_{-1}^{1} fg$. Let U be the subspace of V spanned by $\{1, x^2\}$. Find an orthonormal basis for U and express $2 + x^2$ as a linear combination of that basis.

(2) An $n \times n$ real-valued matrix A is <u>normal</u> if $A^T A = AA^T$.
   (a) Prove that if A is normal, then $\text{null A} = \text{null A}^{\text{T}}$.
   (b) Show this does not have to be true without the normality hypothesis.

(3) Let G be a finite group of order 180 with subgroups H and K. Suppose that $H \cap K$ is not cyclic and $|H| = 36$. Suppose further that $K \not\subseteq H$ and that K fails to contain an element of order 2. What is $|K|$?

(4) An isomorphism from a group G onto itself is called an automorphism of G. You may assume that the set of all automorphisms of G, denoted Aut(G), is a group under composition.
   (a) Let $a \in G$. Prove that the mapping $\phi_a(x) = axa^{-1}$ is an automorphism of G.
   (b) Prove that the set $\text{Inn}(G) = \{\phi_a \mid a \in G\}$ is a normal subgroup of Aut(G).

(5) If R is a commutative ring, prove that $R[x]/\langle x\rangle \cong R$.