Algebra Qualifying Exam
June 7, 2008

Do all five problems.

1. Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

2. Let p be prime. Show that $\{(px, y) \mid x, y \in \mathbb{Z}\}$ is a maximal ideal of $\mathbb{Z} \oplus \mathbb{Z}$.

3. Let V be a vector space over a field F and let $T : V \longrightarrow V$ be a linear operator. Fix a nonzero vector $v \in V$ and define

$$W = \text{span}\{v, T(v), T^2(v), T^3(v), \ldots\}.$$

   (a) Prove that W is T-invariant.

   (b) If $\dim(V)$ is finite and $\dim(W) = k$, prove that

$$\mathcal{B} = \{v, T(v), \ldots, T^{k-1}(v)\}$$

   is a basis for W.

   (c) If $k = 4$, find the matrix of T_W with respect to $\mathcal{B}$ (where T_W denotes the restriction of T to the subspace W).

4. List, up to isomorphism, all Abelian groups of order 225.

5. Let R be an integral domain with the property that every strictly decreasing chain of ideals $I_1 \supset I_2 \supset I_3 \supset \cdots$ has finite length. Prove that R is a field.
   Hint: If $r \in R$ is nonzero, consider the chain $\langle r \rangle \supset \langle r^2 \rangle \supset \langle r^3 \rangle \supset \cdots$