Algebra Qualifying Exam

June 9, 2007

Do all five problems.

1. Recall that a real $n \times n$ matrix is **orthogonal** if $A^T A = I$ (where A^T denotes the transpose of A.) Let $O(n)$ be the set of all $n \times n$ orthogonal matrices.

   (a) If $A \in O(n)$, prove that $\det(A) = \pm 1$.

   (b) Prove that $O(n)$ is a group under matrix multiplication.

   (c) Prove that the set $SO(n) = \{B \in O(n) \mid \det(B) = 1\}$ is a normal subgroup of $O(n)$.

2. Suppose that R is a commutative ring with unity and $I \subseteq R$ is a proper ideal such that every element of $R - I$ is a unit. Prove that I is the unique maximal ideal of R.

3. Let $\langle\ ,\ \rangle$ be a positive definite inner product on the finite-dimensional, real vector space V (recall that $\langle\ ,\ \rangle$ is **positive definite** if $\langle u, u\rangle \geq 0$ for all $u \in V$ and $\langle u, u\rangle > 0$ whenever $u \neq 0$). If T is a linear operator on V, then T^* denotes the adjoint of T.

   (a) Let $w \in V$. Prove that if $\langle v, w\rangle = 0$ for all $v \in V$, then $w = 0$.

   (b) Let S, T be linear operators on V. Prove that $(ST)^* = T^* S^*$.

   (c) Assume that S and T are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

4. Let $H = \{(a, a) \mid a \in \mathbb{Z}\}$. Prove that $(\mathbb{Z} \times \mathbb{Z})/H$ and $\mathbb{Z}$ are isomorphic groups.

5. Consider the ring $R = \left\{ \begin{bmatrix} a & b \\ b & a+b \end{bmatrix} \middle|\, a, b \in \mathbb{Z}_2 \right\}$ under the usual operations of matrix addition and multiplication. Prove that R is a field with 4 elements.