ALGEBRA QUALIFYING EXAM
June 2005

Do all five problems.

1. Let $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the map given by $S(A) = A + A^{\mathrm{T}}$.
   (a) Prove that S is a linear transformation.
   (b) Find a basis for Null S.
   (c) Find a basis for Range S.

2. Let V be a finite dimensional $\mathbb{R}$-vector space with inner product $\langle \ , \ \rangle$. If $W \subseteq V$ is any subspace, prove that $W \oplus W^{\perp} = V$.

3. Suppose that G is a group and $|G| = pq$ for distinct primes p and q.
   (a) Show that every proper subgroup of G is cyclic.
   (b) Is G necessarily Abelian?

4. Let H and K be normal subgroups of a group G. If $H \cap K = \{e\}$, prove that $xy = yx$ for all $x \in H$ and $y \in K$.

5. Let R be a PID and $p \in R$. Prove that the ideal $\langle p \rangle$ is maximal in R if and only if p is a prime element in R. (Recall that p is a prime in R if p is not a unit and if whenever $p|ab$, then $p|a$ or $p|b$).