Algebra Qualifying Exam

May 27, 2006

Do all five problems.

1. Prove that if $\gcd(m,n) = 1$, then $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.

2. Let N be a finitely generated normal subgroup of a group G. If G/N is finitely generated, prove that G is finitely generated.

3. Let $\mathcal{P}_2(\mathbb{R})$ be the space of polynomials with real coefficients and degree less than or equal to 2, with inner product defined by
$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$
for all $f, g \in \mathcal{P}_2(\mathbb{R})$. If $L : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$ is the linear mapping defined by $L(f) = f(0) + f'(0)$, find $g \in \mathcal{P}_2(\mathbb{R})$ such that $L(f) = \langle f, g \rangle$ for all $f \in \mathcal{P}_2(\mathbb{R})$.

   Hint: The set $\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3})\}$ is an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ with respect to $\langle\ ,\ \rangle$.

4. Let F be a field. Prove that $F[\,x\,]$ is a principal ideal domain.

5. Let V be the vector space of 2×2 matrices with real entries and define $T : V \longrightarrow \mathbb{R}^2$ by
$$T(A) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
   (a) Prove that T is a linear transformation.
   (b) Find a basis for $\ker(T)$.