Algebra Qualifying Exam

March 15, 2008

Do all five problems.

1. Let G be a group with center $Z(G)$.

   (a) Prove: If $G/Z(G)$ is cyclic, then G is abelian.

   (b) Prove: If p is prime and G is a nonabelian group of order p^3, then $|Z(G)| = 1$ or $|Z(G)| = p$.

2. Given ideals I, J of a ring R, their **ideal quotient** is the set

$$(I : J) = \{x \in R \,|\, xJ \subseteq I\}.$$

   Prove that $(I : J)$ is an ideal of R.

3. Let $\phi : \mathbb{Q} \longrightarrow \mathbb{Q}$ be a ring automorphism.

   (a) Prove that $\phi(0) = 0$ and $\phi(1) = 1$.

   (b) Prove that $\phi(x) = x$ for all $x \in \mathbb{Q}$.

4. Let $\phi : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_4$ be a group homomorphism.

   (a) Show that $(4, 0, 0) \in \ker(\phi)$.

   (b) Prove that ϕ is not surjective.

5. (a) State the Cayley-Hamilton Theorem.

   (b) Use the Cayley-Hamilton Theorem to compute A^8, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$