Algebra Qualifying Exam

March 17, 2007

Do all five problems.

1. Let G be a group and let $\phi : G \longrightarrow G$ be defined by $\phi(g) = g^2$ for all $g \in G$.

   (a) Prove that ϕ is a homomorphism if and only if G is abelian.

   (b) Prove: If G is a finite abelian group of odd order, then ϕ is one-to-one.

2. Let $\mathcal{M}_2(\mathbb{R})$ be the set of 2×2 matrices with real entries and consider the linear mapping $L : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ defined by $L(A) = A - A^T$ for all $A \in \mathcal{M}_2(\mathbb{R})$.

   (a) Find the matrix of L with respect to the standard ordered basis

   $$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

   for $\mathcal{M}_2(\mathbb{R})$.

   (b) Determine whether L is diagonal or not.

3. Prove that $\mathbb{R}[\,x\,]/\langle x^2 + 1 \rangle$ and $\mathbb{C}$ are isomorphic rings.

4. Let $\mathbb{Z}$ denote the additive group of integers. Prove that for every nonzero integer k there exists a finite group G_k and a surjective homomorphism $\phi : \mathbb{Z} \longrightarrow G_k$ such that $\phi(k) \neq e$ (where e denotes the identity in G_k).

5. Let V be an n-dimensional $\mathbb{R}$-vector space and let $T : V \longrightarrow V$ be a linear transformation such that $\ker(T) = \operatorname{im}(T)$.

   (a) Find the eigenvalues of T.

   (b) What is the characteristic polynomial of T?