Algebra Qualifying Exam
March 11, 2006

Do all five problems.

1. Let $\phi : G \longrightarrow H$ be a group homomorphism.

   (a) Prove that $\mathrm{Ker}(\phi)$ is a normal subgroup of G.

   (b) Prove the First Isomorphism Theorem (i. e., prove that $G/\mathrm{Ker}(\phi) \simeq \phi(G)$).

2. Let R be a commutative ring with 1. Prove that R is a field if and only if $\{0\}$ is a maximal ideal of R.

3. Let $\mathcal{P}_3(\mathbb{R}) = \{a + bx + cx^2 \,|\, a, b, c \in \mathbb{R}\}$, viewed as an $\mathbb{R}$-vector space with standard ordered basis $\{1, x, x^2\}$. Prove that the linear operator $T : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ satisfying

$$\begin{aligned} T(1) &= 2x + 2x^2 \\ T(x) &= 1 + x - x^2 \\ T(x^2) &= 2x^2 \end{aligned}$$

   is diagonalizable.

4. Let $T : V \longrightarrow V$ be a linear operator on a vector space V over a field F where $\dim(V) = 2$. Prove that either there is an $x \in V$ such that $\{x, T(x)\}$ is a basis for V or there is a $c \in F$ such that $T = cI$ (where I is the identity transformation on V).

5. Let G be a group and suppose $a, b \in G$ are such that $|a| = 12$, $|b| = 22$ and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$. Prove that $a^6 = b^{11}$.