

## ALGEBRA QUALIFYING EXAM

March 2005

Do all five problems.

1. Recall that  $\mathcal{P}_3(\mathbb{R})$  is the vector space of all polynomials of degree less than or equal to three with real coefficients, and  $M_2(\mathbb{R})$  is the vector space of all  $2 \times 2$  matrices with real coefficients. Let  $T : \mathcal{P}_3(\mathbb{R}) \rightarrow M_2(\mathbb{R})$  be the linear transformation defined by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a - b & b + c \\ c + d & a - d \end{bmatrix}.$$

- (a) Find a basis for  $\ker(T)$ .
  - (b) Find a basis for  $\text{Range}(T)$ .
2. Let  $(V, \langle \cdot, \cdot \rangle)$  be a real inner product space and let  $T$  be a symmetric (self-adjoint) linear operator (relative to  $\langle \cdot, \cdot \rangle$ ) on  $V$ . Assume that  $v \in V$  is an eigenvector of  $T$  with corresponding eigenvalue  $\lambda \neq 0$ .
    - (a) Prove that if  $w \in v^\perp$ , then  $T(w) \in v^\perp$ .
    - (b) Prove that if  $W$  is a  $T$ -invariant subspace, then  $W^\perp$  is a  $T$ -invariant subspace.
  3. Recall that given a group  $G$ , we denote the center of  $G$  by  $Z(G)$  and the group of inner automorphisms of  $G$  by  $\text{Inn}(G)$ .
    - (a) Prove that  $Z(G) \trianglelefteq G$ .
    - (b) Prove that  $G/Z(G) \cong \text{Inn}(G)$ .
  4. A ring  $R$  is called a Boolean ring if  $a^2 = a$  for all  $a \in R$ . Prove that every Boolean ring  $R$  is commutative. Hint: Prove that  $x = -x$  for all  $x \in R$ , then consider  $(a + b)^2$ .
  5. Suppose that  $R$  is a ring and  $I$  is an ideal in  $R$ . Show that  $R/I$  is an integral domain if and only if  $I$  is a prime ideal.